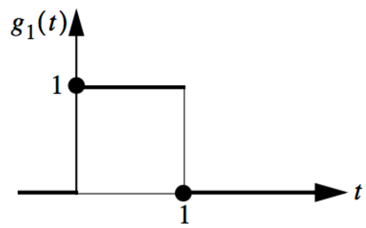
ME-221
SOLUTIONS FOR PROBLEM SET 5

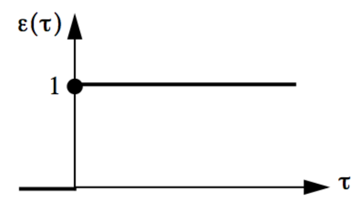
Problem 1

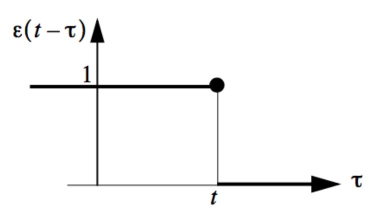
a) $g_1(t) = \varepsilon(t) - \varepsilon(t-1)$



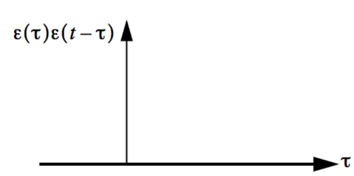
b)

1. $\varepsilon(t) * \varepsilon(t) = \int_{-\infty}^{\infty} \varepsilon(\tau)\varepsilon(t-\tau)d\tau$

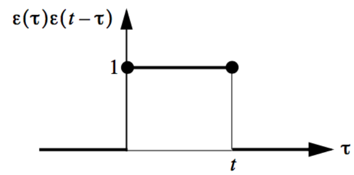




→ For $t < 0$:



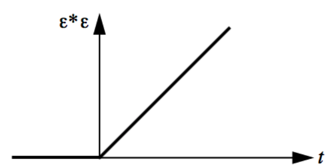
→ For $t \geq 0$:



As a result,

$\varepsilon(t) * \varepsilon(t) = 0$ for $t < 0$

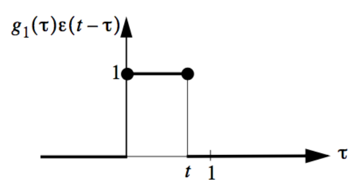
$\varepsilon(t) * \varepsilon(t) = \int_0^t 1 d\tau = t$ for $t \geq 0$



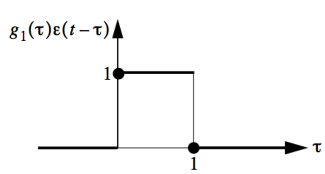
2. $g_1(t) * \varepsilon(t) = \int_{-\infty}^{\infty} g_1(\tau)\varepsilon(t-\tau)d\tau$

→ For $t < 0$: $g_1(\tau)\varepsilon(t-\tau) = 0$

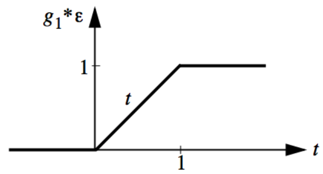
→ For $0 \leq t < 1$:



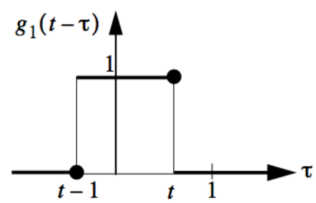
→ For $t \geq 1$:



After integration we obtain:

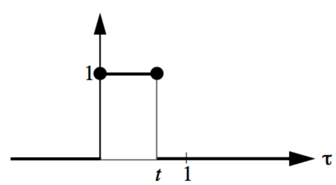


3. $g_1(t) * g_1(t) = \int_{-\infty}^{\infty} g_1(\tau)g_1(t-\tau)d\tau$

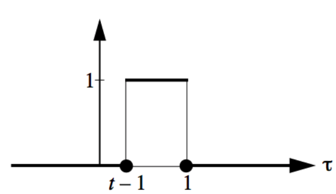


$\rightarrow$ For $t < 0$: $\quad g_1(\tau)g_1(t-\tau) = 0$

$\rightarrow$ For $0 \leq t < 1$:

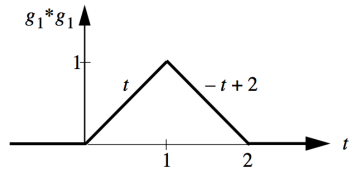


$\rightarrow$ For $1 \leq t < 2$:



$\rightarrow$ For $t \geq 2$: $\quad g_1(\tau)g_1(t-\tau) = 0$

After integration, we obtain:



Problem 2

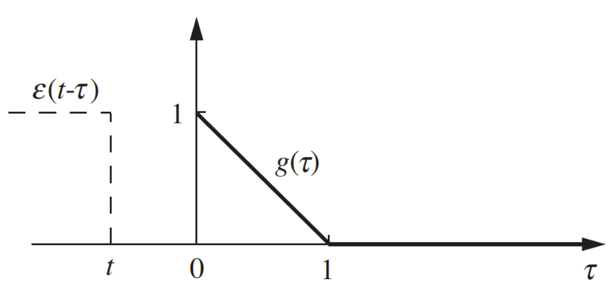
The output of the LTI system is given by $y(t) = g(t) * u(t)$:

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^t e^{-(t-\tau)}e^{-3\tau}d\tau = e^{-t}\int_0^t e^{-2\tau}d\tau =$$
$$= -\frac{1}{2}e^{-t}\left[e^{-2\tau}\big|_0^t\right] = -\frac{1}{2}e^{-t}\left[e^{-2t}-1\right] = -\frac{1}{2}e^{-3t} + \frac{1}{2}e^{-t}$$

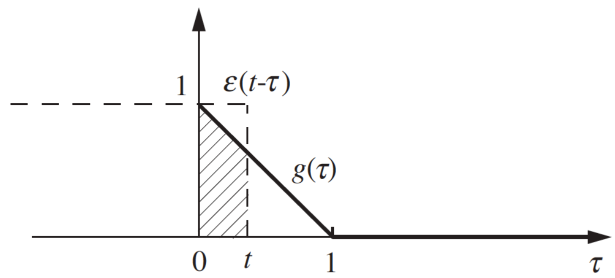
Problem 3

The output is the convolution $y(t) = g(t) * u(t) = g(t) * \varepsilon(t)$.

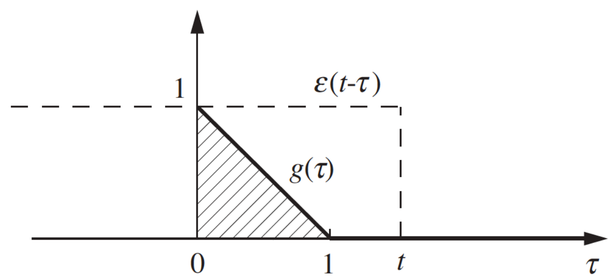
$\rightarrow$ For $t < 0$, $y(t) = 0$



$\rightarrow$ For $0 \leq t < 1$, $y(t) = t - \dfrac{t^2}{2}$

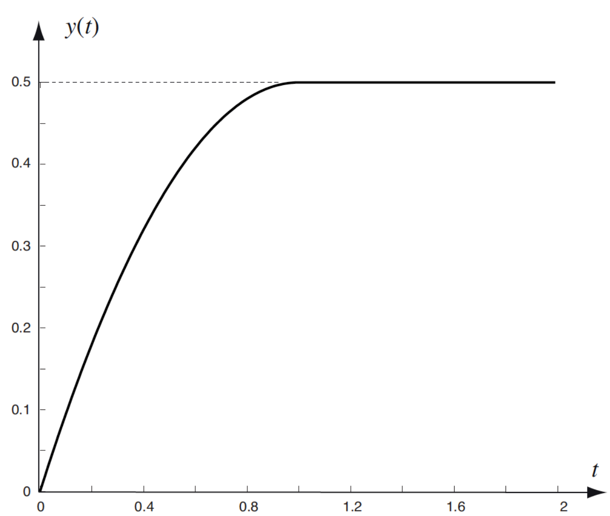


$\rightarrow$ For $t \geq 1$, $y(t) = \dfrac{1}{2}$



The output of the system is given by:

$$y(t) = \begin{cases} 0 & t < 0 \\ t - \dfrac{t^2}{2} & 0 \leq t < 1 \\ \dfrac{1}{2} & t \geq 1 \end{cases}$$



Problem 4

The impulse response of the system can be calculated from the unit step response of the same system by differentiation:

$$g(t) = \frac{ds(t)}{dt}$$

The unit step response is given by $s(t) = (1 - e^{-4t})\varepsilon(t)$ and, thus, $g(t) = 4e^{-4t}\varepsilon(t)$. To be more specific:

$$g(t) = \delta(t) + 4e^{-4t}\varepsilon(t) - e^{-4t}\delta(t) = 4e^{-4t}\varepsilon(t)$$

where $e^{-4t}\delta(t) = \delta(t)$, because of property: $f(x)\delta(x-a) = f(a)\delta(x-a)$, in this case $a = 0$.

The input is given by $u(t) = \varepsilon(t-1) - \varepsilon(t-2)$.

The output of the LTI system is given by $y(t) = g(t) * u(t)$:

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = 4\int_0^t e^{-4(t-\tau)}\varepsilon(\tau-1)d\tau - 4\int_0^t e^{-4(t-\tau)}\varepsilon(\tau-2)d\tau$$

→ For $t < 1$, $y(t) = 0$ as both $\varepsilon(\tau-1)$ and $\varepsilon(\tau-2)$ terms are zero.

→ For $1 \le t < 2$, $\varepsilon(\tau-2) = 0$ but $\varepsilon(\tau-1) = 1$. Thus,

$$y(t) = 4\int_1^t e^{-4(t-\tau)}\varepsilon(\tau-1)d\tau = 1 - e^{4-4t}$$

→ For $t \ge 2$, both $\varepsilon(\tau-2)$ and $\varepsilon(\tau-2)$ are unity. Thus,

$$y(t) = 4\int_1^2 e^{-4(t-\tau)}d\tau = e^{-4t}(e^8 - e^4)$$

There is an alternative solution to this problem. Given the input $u(t)$, which is the sum of two time-delayed step functions:

$$u_1(t) = \varepsilon(t-1)$$
$$u_2(t) = \varepsilon(t-2)$$
$$u(t) = \varepsilon(t-1) - \varepsilon(t-2) = u_1(t) - u_2(t)$$

and the unit step response (the output of the system to an undelayed step input) :

$$s(t) = \left(1 - e^{-4t}\right)\varepsilon(t)$$

Using the linearity property of the convolution operator, we will get that the total response to the input is the sum of the two delayed step responses:

$$\begin{aligned} y_1(t) &= s(t-1)\varepsilon(t-1) \\ y_2(t) &= s(t-2)\varepsilon(t-2) \\ y(t) &= y_1(t) - y_2(t) \\ &= s(t-1)\varepsilon(t-1) - s(t-2)\varepsilon(t-2) \\ &= (1-e^{-4(t-1)})\varepsilon(t-1) - (1-e^{-4(t-2)})\varepsilon(t-2) \end{aligned}$$

Analyzing the conditions:

$$y(t) = \begin{cases} 0, & t < 1 \\ 1 - e^{4-4t}, & 1 \leq t < 2 \\ e^{-4t}\left(e^8 - e^4\right), & t \geq 2 \end{cases} \tag{1}$$